

2.1

Inductive Reasoning

In this lesson you will

- Learn how **inductive reasoning** is used in science and mathematics
- Use inductive reasoning to make **conjectures** about sequences of numbers and shapes

Inductive reasoning is the process of observing data, recognizing patterns, and making generalizations based on those patterns. You probably use inductive reasoning all the time without realizing it. For example, suppose your history teacher likes to give “surprise” quizzes. You notice that, for the first four chapters of the book, she gave a quiz the day after she covered the third lesson. Based on the pattern in your observations, you might generalize that you will have a quiz after the third lesson of every chapter. A generalization based on inductive reasoning is called a **conjecture**.

Example A in your book gives an example of how inductive reasoning is used in science. Here is another example.

EXAMPLE A

In physics class, Dante’s group dropped a ball from different heights and measured the height of the first bounce. They recorded their results in this table.

Drop height (cm)	120	100	160	40	200	80
First-bounce height (cm)	90	74	122	30	152	59

Make a conjecture based on their findings. Then predict the first-bounce height for a drop height of 280 cm.

► **Solution**

If you divide each first-bounce height by the corresponding drop height, you get the following results: 0.75, 0.74, 0.7625, 0.75, 0.76, 0.7375. Based on these results, you could make this conjecture: “For this ball, the height of the first bounce will always be about 75% of the drop height.”

Based on this conjecture, the first-bounce height for a drop height of 280 cm would be about $280 \cdot 0.75$, or 210 cm.

Example B in your book illustrates how inductive reasoning can be used to make a conjecture about a number sequence. Here is another example.

EXAMPLE B

Consider the sequence

10, 7, 9, 6, 8, 5, 7, . . .

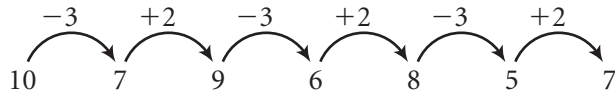
Make a conjecture about the rule for generating the sequence. Then find the next three terms.

(continued)

Lesson 2.1 • Inductive Reasoning (continued)

► **Solution**

Look at how the numbers change from term to term.



The 1st term in the sequence is 10. You subtract 3 to get the 2nd term. Then you add 2 to get the 3rd term. You continue alternating between subtracting 3 and adding 2 to generate the remaining terms. The next three terms are 4, 6, and 3.

In the investigation you look at a pattern in a sequence of shapes.

Investigation: Shape Shifters

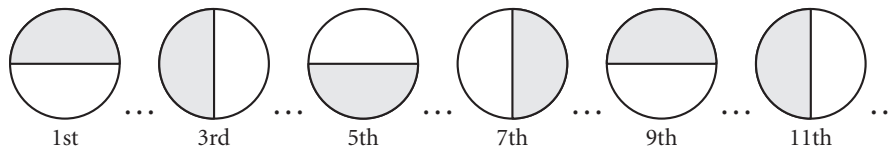
Look at the sequence of shapes in the investigation in your book. Complete each step of the investigation. Below are hints for each step if you need them.

Step 1 Are the shapes the same or different? How does the shaded portion of the shape change from one odd-numbered shape to the next?

Step 2 First, focus on the polygon shape. Does the polygon change from one even-numbered shape to the next? If so, how does it change? Second, focus on the small circles inside the shape. How do these circles change from one even-numbered shape to the next?

Step 3 The next shape is the 7th shape. Because it is an odd-numbered shape, use the patterns you described in Step 1 to figure out what it will look like. The 8th shape is an even-numbered shape, so it should follow the patterns you described in Step 2.

Step 4 Notice that the odd-numbered shapes go through a cycle that repeats every eight terms. So, for example, the 1st, 9th, and 17th shapes look the same; the 3rd, 11th, and 19th shapes look the same; and so on. Use this idea to figure out what the 25th shape looks like.



Step 5 How many sides does the 2nd shape have? The 4th shape? The 6th shape? The n th shape? How many sides will the 30th shape have? How will the dots be arranged on the 30th shape?

Read the text following the investigation in your book.

2.2

Finding the n th Term

In this lesson you will

- Learn how to write **function rules** for number sequences with a constant difference
- Write a rule to describe a geometric pattern
- Learn why a rule for a sequence with a constant difference is called a **linear function**

Consider the sequence 20, 27, 34, 41, 48, 55, 62, Notice that the difference between any two consecutive terms is 7. We say that this sequence has a *constant difference* of 7. To find the next two terms in the sequence, you could add 7 to the last term to get 69, and then add 7 to 69 to get 76. But what if you wanted to find the 200th term? It would take a long time to list all the terms. If you could find a rule for calculating the n th term of the sequence for any number n , you could find the 200th term without having to list all the terms before it. This rule is called the **function rule**. In the investigation you will learn a method for writing a rule for any sequence that has a constant difference.

Investigation: Finding the Rule

Copy and complete each table in Step 1 of the investigation. Then find the difference between consecutive values. If the difference is constant, look for a connection between the difference and the rule.

Here is the completed table for part c. Notice that the values have a constant difference of -2 , which is equal to the coefficient of n in the rule $-2n + 5$.

n	1	2	3	4	5	6	7	8
$-2n + 5$	3	1	-1	-3	-5	-7	-9	-11

For each table, you should have found a constant difference and observed that the constant difference is equal to the coefficient of n in the rule. If you didn't, go back and check your work. In general, if the difference between consecutive terms in a sequence is a constant value a , then the coefficient of n in the rule is a .

Now return to the sequence 20, 27, 34, 41, 48, 55, 62, . . . from the beginning of the lesson. You can organize the terms in a table.

Term n	1	2	3	4	5	6	7	...	200
Value $f(n)$	20	27	34	41	48	55	62	...	

The constant difference for this sequence is 7, so you know that part of the rule is $7n$. The value of the first term ($n = 1$) of the sequence is 20. Substituting 1 for n in $7n$ gives $7(1) = 7$. To get 20, you need to add 13. So, the rule is $7n + 13$.

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Lesson 2.2 • Finding the n th Term (continued)

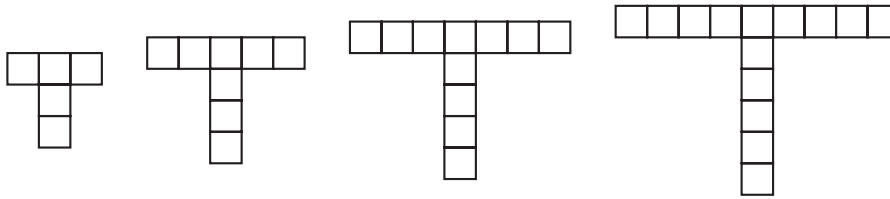
Check this rule by trying it for other terms of the sequence. For example, when $n = 4$, $7n + 13 = 28 + 13 = 41$, which is the 4th term in the sequence.

You can use the rule to find the 200th term in the sequence. The 200th term is $7(200) + 13$, or 1413.

To get more practice writing rules for patterns, work through Examples A and B in your book. Below is another example.

EXAMPLE

If the pattern of T-shapes continues, how many squares will be in the 100th T-shape?



► Solution

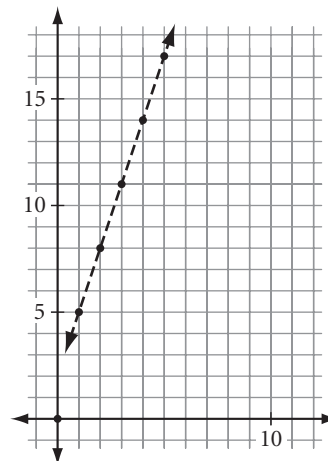
Make a table showing the number of squares in each of the T-shapes shown.

T-shape	1	2	3	4	...	n	...	100
Number of squares	5	8	11	14	

Now try to find a rule for the number of squares in the n th T-shape. Because the constant difference is 3, the rule is of the form $3n + c$. Because the number of squares in the first shape ($n = 1$) is 5, $c = 2$. The rule is $3n + 2$. Therefore, there are $3(100) + 2$, or 302, squares in the 100th T-shape.

In the T-shape example, the process of looking at patterns and generalizing a rule for the n th shape is inductive reasoning. The rule found in this example is called a **linear function**. A linear function is a rule that generates a sequence with a constant difference.

To see why the rule in the T-shape example is a linear function, you can graph the value for the sequence as ordered pairs of the form (*term number*, *value*) on the coordinate plane. The points on your graph will be collinear, because each time the term number increases by 1, the value goes up by the constant difference 3. The line $y = 3x + 2$ passes through the points.



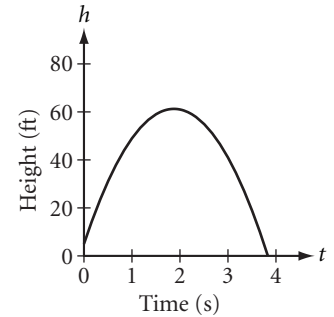
2.3

Mathematical Modeling

In this lesson you will

- Attempt to solve a problem by **acting it out**
- Create a **mathematical model** for a problem
- Learn about **triangular numbers** and the formula for generating them

When you represent a situation with a graph, diagram, or equation, you are creating a **mathematical model**. Suppose you throw a ball straight up into the air with an initial velocity of 60 ft/s. You may recall from algebra that if you release the ball from a height of 5 ft, then the height h of the ball after t seconds can be modeled with the equation and graph shown at right. Once you have created a model, you can use it to make predictions. For example, you could use the equation or graph to predict the height of the ball after 2 seconds or to predict when the ball will hit the ground.



$$h = -16t^2 + 60t + 5$$

In the investigation you will solve a problem by creating mathematical models.

Investigation: Party Handshakes

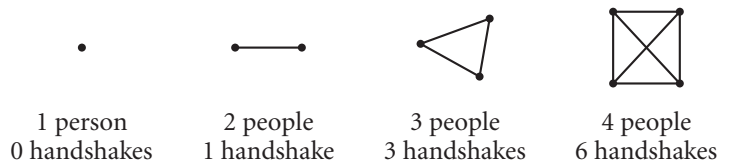
If each of the 30 people at a party shook hands with everyone else, how many handshakes were there altogether?

If you can gather a group of four people, act out the problem for “parties” of one, two, three, and four people and record your results in a table. Your table should look like this.

People	1	2	3	4	...	30
Handshakes	0	1	3	6	...	

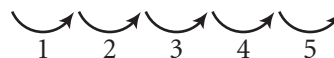
Can you generalize from your table to find the number of handshakes for 30 people? It would certainly help if you had more data. However, gathering many people to act out the problem is not very practical. You could instead try using a mathematical model.

Model the problem by using points to represent people and line segments connecting the points to represent handshakes.



Record your results in a table. This table gives the results for up to six people, but you may want to find results for larger groups of people to help you find a pattern.

Number of points (people)	1	2	3	4	5	6	...	n	...	30
Number of segments (handshakes)	0	1	3	6	10	15	



Notice that the pattern does not have a constant difference, so the rule is not a linear function.

(continued)

Lesson 2.3 • Mathematical Modeling (continued)

Read the dialogue between Erin and Stephanie in your book. According to Erin and Stephanie’s line of reasoning, the diagram with 5 points should have 4 segments per point, so the total number of segments should be $\frac{5 \cdot 4}{2}$, or 10. This matches the data in the table.

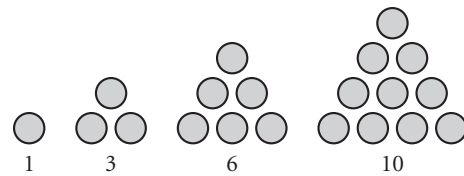
Copy and complete the table given in Step 6 in your book. Make sure you can answer these questions about the expressions for the number of handshakes.

- What does the larger of the two factors in each numerator represent?
- What does the smaller factor represent?
- Why is the product of the factors divided by 2?

You should find that the rule $\frac{n(n-1)}{2}$ models the number of handshakes for a group of n people. So, for 30 people, there would be $\frac{30 \cdot 29}{2}$, or 435, handshakes.

The numbers in the pattern in the investigation are called **triangular numbers** because you can represent them with a triangular pattern of points.

Read the text after the investigation in your book, which shows how you can derive a formula for the triangular numbers using **rectangular numbers**. As expected, the formula is the same one you found in the investigation. Here is another example of a handshake problem.

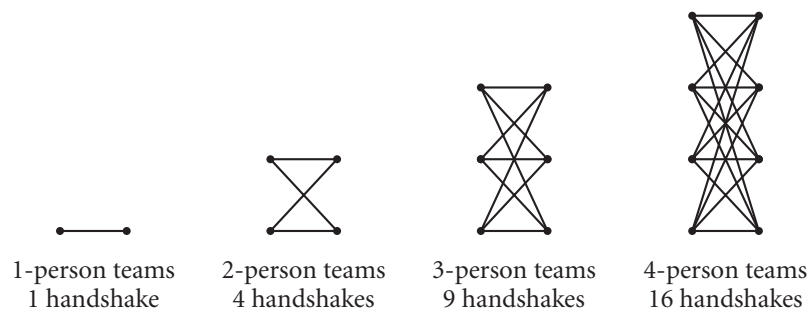


EXAMPLE

Before a soccer game, each of the 11 players on one team shook hands with each player on the other team. How many handshakes were there?

► Solution

Draw diagrams to represent this situation for teams of one, two, three, and four players, and record the results in a table. (Keep in mind that the players do *not* shake hands with members of their own team.)



Players per team	1	2	3	4	...	n	...	11
Number of handshakes	1	4	9	16	

Do you see a pattern? In each case, the number of handshakes is the square of the number of people on each team. The rule for the number of handshakes between two n -player teams is n^2 . So, for the 11-player soccer teams, there were 11^2 , or 121, handshakes.

2.4

Deductive Reasoning

In this lesson you will

- Learn about **deductive reasoning**
- Use deductive reasoning to justify the steps in the solution of an equation
- Use a deductive argument to explain why a geometric conjecture is true

In Lessons 2.1–2.3, you used inductive reasoning to make conjectures based on observed patterns. To explain *why* a conjecture is true, you need to use *deductive* reasoning. **Deductive reasoning** is the process of showing that certain statements follow logically from accepted facts. Read about deductive reasoning on page 114 of your book.

When you give a reason for each step in the process of solving an equation, you are using deductive reasoning. Example A in your book shows the steps involved in solving a particular algebraic equation. Here is another example.

EXAMPLE Solve the equation for x . Give a reason for each step in the solution process.

$$5x^2 + 19x - 45 = 5x(x + 2)$$

► **Solution**

$$5x^2 + 19x - 45 = 5x(x + 2) \quad \text{The original equation.}$$

$$5x^2 + 19x - 45 = 5x^2 + 10x \quad \text{Distribute.}$$

$$19x - 45 = 10x \quad \text{Subtract } 5x^2 \text{ from both sides.}$$

$$-45 = -9x \quad \text{Subtract } 19x \text{ from both sides.}$$

$$5 = x \quad \text{Divide both sides by } -9.$$

Read Example B in your book carefully. It shows three examples of a ray bisecting an obtuse angle. In each case, the two newly formed congruent angles are acute. From these examples, *inductive reasoning* is used to form the following conjecture.

If an obtuse angle is bisected, then the two newly formed congruent angles are acute.

Once the conjecture is stated, a *deductive argument* is used to show it is true. Notice that, by using a variable, m , to represent the measure of an obtuse angle, the argument shows that the conjecture is true for *any* obtuse angle. As another example of using a deductive argument to explain why a conjecture is true, consider the three obtuse angles in Example B of your book with their bisector, \overrightarrow{AC} . This time make a conjecture which states the minimum angle measure for each of the newly formed congruent angles. Looking again at the measures of the new angles— 60° , 79° , and 46° —you notice that each is larger than 45° . So, one possible conjecture is stated here.

Conjecture: If an obtuse angle is bisected, then the two newly formed congruent angles have measures greater than 45° .

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Lesson 2.4 • Deductive Reasoning (continued)

Complete the following deductive argument to show that this conjecture is true.

Deductive Argument: Let a be the measure of an obtuse angle.

$$a > \underline{\hspace{2cm}}$$

When an angle is bisected, the newly formed angles each measure half the original angle.

$$\frac{1}{2}a > \underline{\hspace{2cm}}$$

Therefore, the newly formed congruent angles have measures greater than 45° .

In the investigation you will use inductive reasoning to form a conjecture and deductive reasoning to explain why it is true.

Investigation: Overlapping Segments

Look at the two diagrams at the beginning of the investigation. In each diagram, $\overline{AB} \cong \overline{CD}$.

For each diagram, find the lengths of \overline{AC} and \overline{BD} . What do you notice? You should find that in each case $\overline{AC} \cong \overline{BD}$.

Now draw your own segment \overline{AD} . Place points B and C on the segment so that $\overline{AB} \cong \overline{CD}$ and B is closer to point A than to point D . Measure \overline{AC} and \overline{BD} . You should find that, as in the diagrams in the investigation, $\overline{AC} \cong \overline{BD}$.

Use your findings to complete the conjecture that is started in the book in Step 4. Your completed conjecture should be similar to this one:

If \overline{AD} has points A , B , C , and D in that order with $\overline{AB} \cong \overline{CD}$, then the overlapping segments \overline{AC} and \overline{BD} are congruent (that is, $\overline{AC} \cong \overline{BD}$).

This conjecture is known as the overlapping segments property. Now write a deductive argument to explain why the conjecture is true. (Hint: Use the facts that \overline{BC} is a part of both \overline{AC} and \overline{BD} and that the other parts of \overline{AC} and \overline{BD} —namely, \overline{AB} and \overline{CD} —are congruent.) After you have written your own deductive argument, compare it to the one below.

Deductive Argument: Because $\overline{AB} \cong \overline{CD}$, you know that

$$AB = CD$$

You can add the same thing to both sides of an equation and the equation remains true.

$$AB + BC = BC + CD$$

Using segment addition, $AB + BC = AC$ and $BC + CD = BD$. By substituting AC for $AB + BC$ and BD for $BC + CD$ in the equation above, you get the following:

$$AC = BD$$

By the definition of congruent segments, $\overline{AC} \cong \overline{BD}$.

Read the text following the investigation in your book.

Angle Relationships

In this lesson you will

- Make a conjecture about angles that form a **linear pair**
- Make and prove a conjecture about pairs of **vertical angles**
- Write the **converse** of an “if-then” statement and determine whether it is true

In this lesson you will use inductive reasoning to discover some geometric relationships involving angles.

Investigation 1: The Linear Pair Conjecture

Repeat Step 1 of Investigation 1 three times, creating three different pairs of linear angles.

You should find that, for each pair of linear angles, the sum of the angle measures is 180° . This discovery leads to the following conjecture.

Linear Pair Conjecture If two angles form a linear pair, then the measures of the angles add up to 180° .

C-1

Keep a list of important conjectures in your notebook. Make a sketch for each conjecture. The Linear Pair Conjecture (C-1) should be the first entry in your list.

Investigation 2: Vertical Angles Conjecture

Follow Steps 1–3 in your book. You should find that the vertical angles are congruent. That is, $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

Now draw a different pair of intersecting lines on patty paper and repeat Steps 2 and 3. Are the vertical angles congruent?

Fill in the conjecture using your work in this investigation.

Vertical Angles Conjecture If two angles are vertical angles, then they are _____.

C-2

You used inductive reasoning to discover the Linear Pair Conjecture and the Vertical Angles Conjecture. The example in your book shows that, if you accept that the Linear Pair Conjecture is true, you can use deductive reasoning to show that the Vertical Angles Conjecture must also be true. Read the example very carefully and make sure you understand each step of the deductive argument.

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Lesson 2.5 • Angle Relationships (continued)

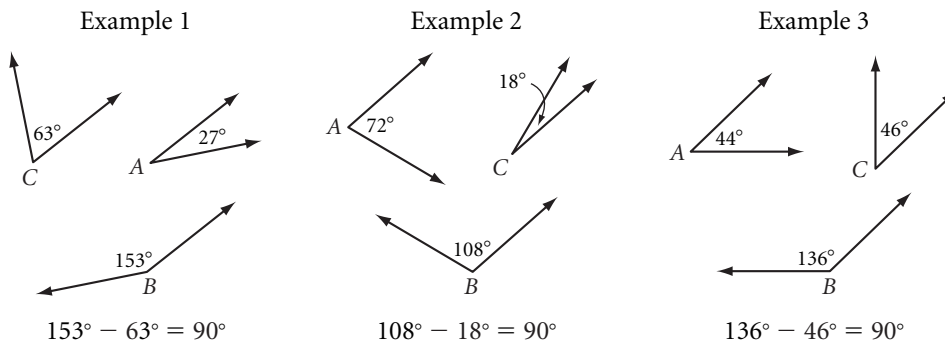
Here is another example to help you write deductive arguments.

EXAMPLE

- a. Use inductive reasoning to complete this conjecture.
 If $\angle B$ is the supplement of an acute angle A and $\angle C$ is the complement of $\angle A$, then $m\angle B - m\angle C = \underline{\hspace{2cm}}$.
- b. Write a deductive argument to show why the conjecture is true.

► Solution

- a. The diagrams below show three examples.



In each case, $m\angle B - m\angle C = 90^\circ$. Based on these examples, the conjecture would be

If $\angle B$ is the supplement of an acute angle A and $\angle C$ is the complement of $\angle A$, then $m\angle B - m\angle C = 90^\circ$.

Deductive Argument: The measures of supplementary angles add up to 180° .

$$m\angle A + m\angle B = 180^\circ$$

The measures of complementary angles add up to 90° .

$$m\angle A + m\angle C = 90^\circ$$

Solve the second equation for $m\angle A$.

$$m\angle A = 90^\circ - m\angle C$$

Substitute $90^\circ - m\angle C$ for $m\angle A$ in the first equation.

$$90^\circ - m\angle C + m\angle B = 180^\circ$$

Subtract 90° from both sides of the equation.

$$-m\angle C + m\angle B = 90^\circ$$

Rewrite the left side of the equation to complete the deductive argument.

$$m\angle B - m\angle C = 90^\circ$$

The Vertical Angles Conjecture states that *if* two angles are vertical angles, *then* they are congruent. When you reverse the “if” and “then” parts of an “if-then” statement, you get the **converse** of the statement. Here is the converse of the Vertical Angles Conjecture.

If two angles are congruent, *then* they are vertical angles.

Is this statement true? The diagram on page 124 of your book shows a *counterexample* to this statement. If you can find even one counterexample, then the statement is false. So, the converse of the Vertical Angles Conjecture is false.

Special Angles on Parallel Lines

In this lesson you will

- Make three conjectures about the angles formed when two parallel lines are intersected by a **transversal**
- Determine whether the converse of each conjecture is true
- Prove one of the conjectures assuming one of the other conjectures is true

A line that intersects two or more coplanar lines is called a **transversal**. In your book, read about the three types of angle pairs formed when a transversal intersects two lines. In the investigation you will look at the angles formed when a transversal intersects two *parallel* lines.

Investigation 1: Which Angles Are Congruent?

Follow the instructions before Step 1 to create parallel lines k and ℓ intersected by transversal m . Number the angles as shown.

Place a piece of patty paper over the set of angles 1, 2, 3, and 4. Copy the two intersecting lines m and k and the four angles onto the patty paper.

Angles 1 and 5 are corresponding angles. Place the tracing of $\angle 1$ over $\angle 5$. How do the angles compare? Repeat this process for the other pairs of corresponding angles ($\angle 3$ and $\angle 7$, $\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$). Use your findings to complete this conjecture.

Corresponding Angles Conjecture, or CA Conjecture If two parallel lines are cut by a transversal, then corresponding angles are _____.

C-3a

Angles 3 and 6 are alternate interior angles. Place the tracing of $\angle 3$ over $\angle 6$. How do the angles compare? Repeat this process for the other pair of alternate interior angles ($\angle 4$ and $\angle 5$), and then complete this conjecture.

Alternate Interior Angles Conjecture, or AIA Conjecture If two parallel lines are cut by a transversal, then alternate interior angles are _____.

C-3b

Angles 1 and 8 are alternate exterior angles. Place the tracing of $\angle 1$ over $\angle 8$. How do the angles compare? Repeat this process for the other pair of alternate exterior angles ($\angle 2$ and $\angle 7$), and then complete this conjecture.

Alternate Exterior Angles Conjecture, or AEA Conjecture If two parallel lines are cut by a transversal, then alternate exterior angles are _____.

C-3c

The three conjectures above can be combined to create the Parallel Lines Conjecture.

Parallel Lines Conjecture If two parallel lines are cut by a transversal, then corresponding angles are _____, alternate interior angles are _____, and alternate exterior angles are _____.

C-3

(continued)

Lesson 2.6 • Special Angles on Parallel Lines (continued)

Now draw two lines that are *not* parallel and a transversal cutting both lines. Use the same process you used above to compare corresponding angles, alternate interior angles, and alternate exterior angles. Do the conjectures work for nonparallel lines?

Investigation 2: Is the Converse True?

In this investigation you'll consider the converse of the Corresponding Angles Conjecture: If two lines are cut by a transversal to form a pair of congruent corresponding angles, then the lines are parallel. Do you think this statement is true? Investigate by following Step 1 in your book.

Now write the converse of each of the other two conjectures. Do you think the converses are true? Investigate by following Step 2 in your book. Then complete the following conjecture.

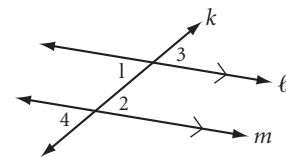
Converse of the Parallel Lines Conjecture If two lines are cut by a transversal to form a pair of congruent corresponding angles, congruent alternate interior angles, or congruent alternate exterior angles, then the lines are _____.

C-4

If you accept any of the three parallel lines conjectures as true, you can use deductive reasoning (and possibly some earlier conjectures) to show that the others are true. The example in your book shows that if you accept the Corresponding Angles and Vertical Angles Conjectures as true, then you can prove that the Alternate Interior Angles Conjecture is true. Read the example carefully. Here is another example.

EXAMPLE

Suppose you assume that the Alternate Interior Angles Conjecture is true. Write a deductive argument showing that the Alternate Exterior Angles Conjecture must be true. (You may assume that the Vertical Angles Conjecture is true.)



► Solution

Deductive Argument: In the diagram, lines ℓ and m are parallel and intersected by transversal k . Because we assume that the alternate interior angles are congruent

$$m\angle 1 = m\angle 2$$

Because we assume the Vertical Angles Conjecture is true

$$m\angle 1 = m\angle 3 \text{ and } m\angle 2 = m\angle 4$$

Substitute $m\angle 3$ for $m\angle 1$ and $m\angle 4$ for $m\angle 2$ in the first equation.

$$m\angle 3 = m\angle 4$$

So, the alternate exterior angles, $\angle 3$ and $\angle 4$, are congruent.

Therefore, if the alternate interior angles are congruent, then the alternate exterior angles are also congruent.